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Traveling Waves in the SIS Generalized Epidemic Model With Infection-Age Structure

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Abstract

In this work, traveling wave solutions (TWS) of a diffusive SIS epidemic model incorporating infection age structure and a general nonlinear incidence function are investigated. Through analysis of the associated differential system, the basic reproduction number R_0 is identified as the fundamental threshold parameter: when $R_0 \leq 1$, disease invasion is prevented and no nontrivial wave emerges, whereas for $R_0 > 1$, the dynamics are characterized by a strictly positive minimal wave speed $c^* > 0$. By employing spectral methods together with rigorously constructed upper and lower solutions, it is demonstrated that monotone waves connecting the disease free and endemic equilibria exist if and only if the propagation speed satisfies $c \geq c^*$. Moreover, such waves are shown not to occur for $0 < c < c^*$. Numerical simulations, formulated through phase plane trajectories and traveling wave profiles, are presented to validate the theoretical prediction of a sharp threshold at $c = c^*$.

Keywords: minimal wave speed (MWS); upper and lower solutions; age structured; basic reproduction number.

1 Introduction

Mathematical models of infectious diseases are at the heart of controlling and understanding epidemics [5, 36]. Of the many models available, the Susceptible-Infectious-Susceptible (SIS) model is most relevant to infections where the individual never acquires long-term immunity upon infection [6, 21]. Traditional SIS models generally assume homogeneous mixing and instantaneous compartmental transitions that may oversimplify disease dynamics [17, 29]. Inserting an infection age structure, where every infected host has a time since infection, gives a more realistic representation of the dynamics of the disease [8, 33]. This enables variation in infectivity during an infection and the ability to model latency periods or stage-specific transmission rates [18, 32]. These models are likely to lead to integro-differential equations, adding to the analysis difficulty but producing a more realistic model of disease transmission [25, 30].

The concept of TWS is employed to account for the dissemination of infectious diseases through space. Traveling wave solutions account for waves of infection spreading across space at a constant speed, spanning disease-free regions to epidemic states [20, 26]. Traveling waves are handy when evaluating the situations under which an infection can settle and hold onto a new territory [28, 34].

In recent years, there has been increasing interest in incorporating both spatial and temporal heterogeneity into epidemiological models [1, 7]. In this case, the incorporation of both spatial spread and infection-age structure leads to more complicated models that better reflect the biological complexity of real epidemics [2, 31]. These models often comprise partial differential equations with integral terms incorporated to capture infection history. In these models, a generalized incidence function is usually employed to include nonlinearities, e.g., transmission saturation due to limited contact rates, behavioral change in response to the epidemic, or resource constraints [9, 37]. Generalized functions offer a more flexible modeling of the transmission process compared to the default bilinear incidence βSI, especially in areas of high-risk or high-density.

Unlike classical SIS traveling wave models that typically assume bilinear incidence and disregard temporal heterogeneity in infectiousness [23, 27], the present study incorporates a nonlinear generalized incidence function together with infection-age structure [22, 24]. Such coupling allows the transmission rate to depend on the age of infection while simultaneously capturing saturation and behavioral feedback, yielding a much more biologically realistic formulation than previous SIS wave models. This model is important for recurrent-infection diseases in which immunity is short-lived or nonexistent, including COVID-19, malaria, gonorrhea, and seasonal influenza. Such pathogens often manifest reinfection and heterogeneous infectiousness, which have motivated the inclusion of generalized transmission operators and age-of-infection structure in wave dynamics models [3, 4].

Recent work has focused on fractional dynamics, information-driven behavioral responses, and hybrid data-driven dynamical systems. For instance, fractional-order epidemic systems in the presence of media influence can produce realistic memory effects in population behavior [15, 16]. These studies give relevance to our generalized SIS framework and position the present traveling-wave analysis in a modern context of heterogeneous and memory-driven epidemic transmission.

The study of TWS in a generalized SIS epidemic model with infection-age structure thus becomes both mathematically challenging and biologically interesting. It provides additional understanding of how spatially structured diseases with time-dependent infectiousness may spread and persist endemically [19, 35]. Such are crucial factors in public health planning, particularly for repeated infections where hosts do not become immune and can be reinfected on many occasions.

Recent studies have extended the classical SIS models by adding spatial diffusion, nonlocal dispersal, to provide more realistic models of disease propagation. The existence and characterization of TWS for such complicated models can be reliant on advanced mathematical techniques [10, 14]. Techniques employed involve the method of constructing upper and lower solutions, spectral analysis, and fixed-point theorems to demonstrate the existence of TWS and establish the minimal wave speed MWS (c^*) for the wave to propagate [12, 13]. Despite these advances, study of TWS of SIS models with infection-age structure and generalized incidence rates remains relatively in its early stages. This is significant as generalized incidence functions can quantify saturation effects and behavioral change in response to infection prevalence. This gap closure is crucial towards more realistic and predictive models for disease spread.

In this paper, we aim to investigate the existence of traveling wave solutions in a spatially structured SIS epidemic model with infection-age structure and general incidence rates. We will use analytical techniques to determine under what conditions such waves exist and to calculate the minimum wave speed c^*. Our work tries to extend the analysis in the existing literature and shed more light on the complex behavior of infectious disease transmission.

In light of the aforementioned issues, we provide the following diffusive epidemic model with a nonlinear incidence rate;

$$\begin{cases} \dfrac{\partial S}{\partial t} = d_1 \Delta S - M\left(S(x,t), H(I(x,b,t))\right) + \displaystyle\int_0^{+\infty} \sigma(b) I(x,b,t)db, \\ \dfrac{\partial I}{\partial t} + \dfrac{\partial I}{\partial b} = d_2 \Delta I - (\sigma(b)) I(x,b,t), \\ I(x,0,t) = M(S(x,t), H(I(x,b,t))), \qquad S(x,0) = S_0(x), \\ I(x,b,0) = I_0(x,b), \end{cases} \tag{1}$$

with $H(I(x,b,t)) = \displaystyle\int_0^{+\infty} \varrho(a) I(x,b,t)da,\ b \in [0,+\infty),\ x \in \mathbb{R}$ and $t > 0$.

The susceptible and infected people at location x and time t, respectively, are represented by $S(x,t)$ and $I(x,b,t)$, while b is a measure of the amount of time spent in the infected compartment. The term $\displaystyle\int_0^{+\infty} \sigma(b) I(x,b,t)db$ represents the flux back to the susceptible class from disease-related deaths. The diffusion coefficients are shown by the positive parameters d_1 and d_2. The decay rate of infected people, which includes recover rate $\sigma(b)$, where $\sigma(b) > 0$ as a function of age a. The infection rate is $\varrho(b)$, and the disease-related recover rate is $\sigma(b)$.

Also, we suppose that $M \in C^2(\mathbb{R}^2)$, and satisfies;

(A) $d_1, d_2, \Lambda > 0$ and $\varrho(b) \in L^1(0,\infty) \bigcap L^\infty(0,\infty)$, $\sigma(b) \in L^\infty_{\text{loc}}(0,\infty)$.

(H) $M(S,0) = M(0,I) = 0$, $\dfrac{\partial M(S,I)}{\partial I} > 0$, $\dfrac{\partial^2 M(S,I)}{\partial I^2} < 0$ and $\dfrac{\partial M(S,I)}{\partial S} > 0$, $\forall\, S, I > 0$.

It clarifies the age-related variation in the survival rate of infected individuals. For $h(b)$, $h(b)\varrho(b) \in L_1(0,+\infty) \bigcap L^\infty(0,+\infty)$ and $\mu(b) + \sigma(b) \in L^1_{loc}(0,+\infty)$, $\varrho(b) \in L^\infty(0,+\infty)$ are assumed. Furthermore, in order to guarantee that the probability of individual existence finally vanishes (that is, the infection duration exceeds the species' physiological survival age), we hypothesis that,

$$\int_0^{+\infty} \sigma(b)db = +\infty.$$

We organize our research in this way to tackle these issues. The traveling wave system is formulated in the following section, and its existence is demonstrated using the upper and lower solution approach. The purpose of Section 3 is to demonstrate that traveling wave solutions do not exist when either $R_0 \leq 1$ or $R_0 > 1$ and $c < c^*$ are true. In Section 4, we additionally use a few graphical representations to bolster the primary theoretical findings. Lastly, a conclusion in Section 5 brings the investigation to a close.

2 Existence of TWS

We first provide R_0 and c^* to demonstrate the existence of traveling wave solutions (TWS). To accomplish this, the system obtained by assuming spatial homogeneity of (1) can be expressed as;

$$\begin{cases} \dfrac{dS}{dt} = d_1\Delta S - M(S(t), H(I(b,t))) + \displaystyle\int_0^{\infty} \sigma(b)I(b,t)da, \\ \dfrac{\partial I}{\partial t} + \dfrac{\partial I}{\partial b} = d_2\Delta I - (\mu(b)+\sigma(b))I(b,t), \\ I(0,t) = M(S(t), H(I(b,t))), \qquad S(0) = S_0, \\ I(b,0) = I_0(b). \end{cases} \tag{2}$$

It is simple to verify that system (2)'s basic reproduction number may be determined by,

$$R_0 = \frac{\partial M(S_0,0)}{\partial I}\int_0^{\infty} \varrho(b)h(b)db.$$

Substituting the variable $i(x,b,t) = \dfrac{I(x,b,t)}{h(b)}$ with $h(b) = \exp^{-\int_0^b (\mu(v)+\sigma(v))dv}$ to system (3) results in,

$$\begin{cases} \dfrac{\partial S}{\partial t} = d_1\Delta S - M(S(x,t), H(i(x,b,t))) + \displaystyle\int_0^{\infty} \sigma(b)h(b)i(x,b,t)da, \\ \dfrac{\partial i}{\partial t} + \dfrac{\partial i}{\partial b} = d_2\Delta i, \\ i(x,0,t) = M(S(x,t), H(i(x,b,t))), \qquad S(x,0) = S_0(x), \\ i(x,a,0) = i_0(x,b) = \dfrac{I_0(x,b)}{h(b)}, \end{cases} \tag{3}$$

with $H(i(x,b,t)) = \int_0^{\infty} \varrho(b)h(b)i(x,b,t)db,\ b\in[0,\infty),\ x\in\mathbb{R}$ and $t>0$.

The profile equations for system (3) still need to be provided. The TWS of (3) achieves

$$S(t,x) = S(\chi), i(t,x,b) = i(\chi,b) \qquad \text{and} \qquad \chi = x + ct,$$

where χ and $c > 0$ represent the coordinate system in motion and the wave propagation speed, respectively. Thus, the following interpretation of system (3) profile equations is possible,

$$\begin{cases} cS'(\chi) = d_1 S''(\chi) - M(S(\chi), H(i(\chi, b))) + \int_0^\infty \sigma(b)h(b)i(\chi, b)db, \\ c\dfrac{\partial i}{\partial \chi} + \dfrac{\partial i}{\partial b} = d_2 \Delta_\chi i, \\ i(\chi, 0) = M(S(\chi), H(i(\chi, b))), \end{cases} \tag{4}$$

where $H(i(\chi, b)) = \int_0^\infty \varrho(b)h(b)i(\chi, b)db,\ b \in [0, \infty)$ and $\chi \in \mathbb{R}$.

Our focus is on finding the solution to system (4) that satisfies the asymptotic boundary conditions listed below:

$$S(-\infty) = S_0, \qquad S(+\infty) = S^*, \qquad i(\pm\infty, b) = 0, \tag{5}$$

where the severity of infectious illnesses is measured by S^*, and $S_0 > S^* \geq 0$.

i. $S_0 = S^*$ indicates that the disease has been effectively managed and that its effects on the general populace can be almost ignored.

ii. $S^* < S_0$ indicates that the disease affects the population as a whole more. More specifically, the number of people lost to sickness is described as $v = S_0 - S^*$.

Finding a solution for $i(\chi, b = \exp^{\chi - \varpi b}$ for both this linearized equation and the second equation of system (4) after linearizing $i(\chi, 0)$ at the disease-free equilibrium $(S_0, 0)$ yields the characteristic equation that follows,

$$1 = \frac{\partial M(S_0, 0)}{\partial i} \int_0^\infty \varrho(b)h(b) \exp^{-\varpi b} db, \tag{6}$$

and

$$X(, c) = d_2{}^2 - c + \varpi.$$

Then, if $R_0 > 1$ indicates the existence of a special constant $c^* > 0$ such that,

$$\frac{\partial M(S_0, 0)}{\partial i} \int_0^\infty \varrho(b)h(b) \exp^{-\varpi^* b} db = 1, \tag{7}$$

where $d_2{}^2 - c = \varpi^*$.

Thus, if and only if $c > 2\sqrt{d_2 \varpi^*} = c^*$, $X(, c^*)$ has real roots, and its lowest real root is represented by,

$$_1 = \frac{c - \sqrt{c^2 - 4d_2\varpi^*}}{2d_2} \in \left(0, \frac{c}{2d_2}\right),$$

and

$$_2 = \frac{c + \sqrt{c^2 - 4d_2\varpi^*}}{2d_2}.$$

After completing a number of procedures, we will now demonstrate the existence of traveling wave solutions for system (4). Additionally, we always assume that $c > c^*$ and $R_0 > 1$. In order to achieve this, we first present a few lemmas that are crucial to demonstrating the primary finding.

2.1 Upper and lower solution

We developed two super and sub solutions of (4) motivated by the following statement:

Lemma 2.1. *The function $i^+(\chi, b) = \exp^{\lambda_1\chi - \varpi^* b}$ satisfies the following equation,*

$$c\frac{\partial i^+}{\partial \chi} + \frac{\partial i^+}{\partial b} = d_2\Delta_\chi i^+, \qquad \frac{\partial M(S_0, 0)}{\partial I}\int_0^\infty \varrho(b)h(b)\exp^{-\varpi^* b} db. \tag{8}$$

Proof. The proof is clear by using the definition of λ_1. □

Lemma 2.2. *Simply $S^+(\chi) = S_0$ satisfies,*

$$-c(S^+)'(\chi) + d_1(S^+)''(\chi) - M(S^+(\chi), H(i^-(\chi, b))) + \int_0^\infty \sigma(b)h(b)i^-(\chi, b)db \leq 0. \tag{9}$$

Proof. Therefore, (9) is satisfied. □

Assume that,

$$\frac{\partial M(S_0, 0)}{\partial I}\int_0^\infty \varrho(b)h(b)i^+(\chi, b)db = \exp^{\lambda_1\chi}.$$

Next, we prove the following lemma.

Lemma 2.3. *The positive equation $S^-(\chi) = \max\left\{1 - Q\exp^{\vartheta\chi}, 0\right\}$ satisfies,*

$$-c(S^-)'(\chi) + d_1(S^-)''(\chi) - M(S^-(\chi), H(i^+(\chi, b))) + \int_0^\infty \sigma(b)h(b)i^+(\chi, b)db \geq 0. \tag{10}$$

Proof. Let $0 < \vartheta < \min\left\{\lambda_1, \frac{c}{d_1}\right\}$. Suppose that $\chi \neq \frac{1}{\vartheta}\ln\frac{1}{Q} := \chi^*$, and we claim that S^- meets (10). In order to demonstrate this assertion, we first assume that $\chi > \chi^*$. This suggests that, in (χ^*, ∞), $S^-(\chi) = 0$, and the inequality is therefore immediately true. If $\chi < \chi^*$, we have $S^-(\chi) = S^0 - Qe^{\vartheta\chi}$. Then, we have

$$\begin{aligned}
&- c(S^-)'(\chi) + d_1(S^+)'' - M(S^-(\chi), H(i^+(\chi, b))) + \int_0^\infty \sigma(b)h(b)i^+(\chi, b)db \\
&\quad \geq cQ\vartheta\exp^{\vartheta\chi} - d_1Q\vartheta^2e^{\vartheta\chi} - \frac{\partial M(S_0, 0)}{\partial I}\int_0^\infty \varrho(b)h(b)i^+(\chi, b)db + \int_0^\infty \sigma(b)h(b)i^+(\chi, b)db, \\
&\quad = Q\exp^{\vartheta\chi}\left[c\vartheta - d_1\vartheta^2\right] - \exp^{\lambda_1\chi} + \int_0^\infty \sigma(b)h(b)\exp^{\lambda_1\chi - \varpi^* b} db, \\
&\quad \geq \exp^{\vartheta\chi}\left[Qc\vartheta - d_1Q\vartheta^2 - \left(\frac{S_0}{Q}\right)^{\frac{\lambda_1 - \vartheta}{\vartheta}} + \int_0^\infty \sigma(b)h(b)\exp^{-\varpi^* b} db\left(\frac{S_0}{Q}\right)^{\frac{\lambda_1}{\vartheta}}\right].
\end{aligned}$$

Here we use,

$$\exp^{\lambda_1\chi} < \left(\frac{S^0}{Q}\right)^{\frac{\lambda_1 - \vartheta}{\vartheta}}, \qquad \text{for} \quad \chi < \chi^*.$$

Keeping $\vartheta Q = 1$, letting $Q \to \infty$ for some $Q > S^0$ large enough and ϑ small enough, we have

$$-c(S^-)'(\chi) + d_1(S^-)''(\chi) - M(S^-(\chi), h(i^+(\chi, b))) + \int_0^\infty \sigma(b)h(b)i^+(\chi, b)db \geq 0.$$

Then, (10) is hold. □

Lemma 2.4. *For all sufficiently small $\gamma < \vartheta$, there is a sufficiently large constant L such that the function,*

$$I^-(\chi, b) = \max\{\exp^{\lambda_1\chi - \varpi^* b} - L\exp^{(\lambda_1+\gamma)\chi - \eta_1 b}, 0\},$$

satisfies,

$$\begin{cases} \dfrac{\partial i^-}{\partial b} = d_2\Delta_\chi i^- - c\dfrac{\partial i^-}{\partial \chi}, \\ i^-(\chi, 0) \leq M\left(S(\chi), \displaystyle\int_0^\infty \varrho(b)h(b)i^-(\chi, b)db\right), \\ \displaystyle\int_0^\infty \varrho(b)h(b)i^-(\chi, b)db = H^-, \end{cases} \tag{11}$$

where $\eta_1 = c(_1 + \gamma) - d_2(_1 + \gamma)^2 > 0$.

Proof. For simplicity, let $k(x) = \dfrac{M(S_0, x)}{x}$ and $\lim_{x\to 0} k(0) = \partial_2 M(S_0, 0)$. It follows from the concavity of M that k is a decreasing function of x, and there exists M_k such that,

$$0 < \left(\partial_2 M(S_0, 0) - \frac{M(S^-(\chi), x)}{x}\right) \leq M_k, \quad \text{for all } x \geq 0.$$

If $\chi > \chi_0(b) := \dfrac{1}{\gamma}[\ln(L) + (\varpi^* - \eta_1)b]$, then $i^-(\chi, b) = 0$, which clearly satisfies (11). Now, we verify the inequality for $\chi < \chi_0(b)$. We write

$$i^-(\chi, 0) - M\left(S^-(\chi), \int_0^\infty \varrho(b)h(b)i^-(\chi, b)db\right) = i^-(\chi, 0) - \frac{M(S^-(\chi), H^-)}{H^-}H^-,$$

where $H^- = \displaystyle\int_0^\infty \varrho(b)h(b)i^-(\chi, b)db$. Substituting $I^-(\chi, b)$, we obtain

$$\begin{aligned} I^-(\chi, 0) - M(S^-, H^-) = \exp^{\lambda_1\chi}&\left[1 - \frac{M(S^-, H^-)}{H^-}\int_0^\infty \varrho(b)h(b)e^{-\varpi^* b}db\right] \\ &- L\exp^{(\lambda_1+\gamma)\chi}\left[1 - \frac{M(S^-, H^-)}{H^-}\int_0^\infty \varrho(b)h(b)e^{-\eta_1 b}db\right]. \end{aligned}$$

Since that $\varpi^* = c - d_2{}^2$ increases monotonically on $(_1, _2)$, we have $\eta_1 > \varpi^*$. It follows from (6), and the existence of $_1$ that,

$$\partial_2 M(S_0, 0)\int_0^\infty \varrho(b)h(b)e^{-\varpi^* b}db = 1,$$

and

$$\partial_2 M(S_0, 0)\int_0^\infty \varrho(b)h(b)e^{-\eta_1 b}db < 1.$$

Hence,

$$\frac{M(S^-,H^-)}{H^-}\int_0^\infty \varrho(b)h(b)e^{-\eta_1 b}db \le \partial_2 M(S_0,0)\int_0^\infty \varrho(b)h(b)e^{-\eta_1 b}db < 1.$$

Also, we have

$$\begin{aligned}1-\frac{M(S^-,H^-)}{H^-}\int_0^\infty \varrho(b)h(b)e^{-\varpi^* b}db &= \left(\partial_2 M(S_0,0)-\frac{M(S^-,H^-)}{H^-}\right)\int_0^\infty \varrho(b)h(b)e^{-\varpi^* b}db\\ &\le M_k\int_0^\infty \varrho(b)h(b)e^{-\varpi^* b}db.\end{aligned}$$

Therefore,

$$\begin{aligned}&i^-(\chi,0)-M(S^-,H^-)\\ &\quad\le \exp^{\lambda_1\chi} M_k\int_0^\infty \varrho(b)h(b)e^{-\varpi^* b}db - L\exp^{(\lambda_1+\gamma)\chi}\left(1-\partial_2 M(S_0,0)\int_0^\infty \varrho(b)h(b)e^{-\eta_1 b}db\right).\end{aligned}$$

By choosing L sufficiently large, we ensure,

$$i^-(\chi,0)-M(S^-,H^-)\le 0.$$

Thus, the inequality in (11) holds. The proof is complete. □

2.2 Truncated problem

We primarily examine the following issue with the limited interval $(-\tau,\tau)$ in this subsection;

$$\begin{cases} cS''=d_1S''-M(S(\chi),\int_0^\infty \varrho(b)h(b)i(\chi,b))\,db+\int_0^\infty \sigma(b)i(\chi,b)db, & \chi\in(-\tau,\tau),\\ \dfrac{\partial i}{\partial b}=d_2\Delta_\chi i-c\dfrac{\partial i}{\partial \chi}, & \chi\in(-\tau,\tau),\ b\in(0,\infty),\\ i(\chi,0)=M(S(\chi),\int_0^\infty \varrho(b)h(b)i(\chi,b))\,db, & \chi\in[-\tau,\tau],\\ S(\pm\tau)=S^-(\pm\tau), & i(\pm\tau,b)=i^-(\pm\tau,b),\ b\in(0,\infty).\end{cases}\tag{12}$$

We will demonstrate that issue (12) has a classical solution by using the upper and lower-solutions defined by Schauder's fixed-point theorem in the aforementioned part. More specifically, we get the following outcome.

Theorem 2.1. *Suppose $\tau_0\ge \frac{1}{\gamma}\ln L$ is a sufficiently large constant. Then, the boundary value problem (12) admits a classical solution $(S(\chi),i(\chi,b))$ satisfying for any $\tau>\tau_0$,*

$$\begin{aligned}&S^-(\chi)\le S(\chi)\le S_0, && \text{for }\ \chi\in(-\tau,\tau),\\ &i^-(\chi,b)\le i(\chi,b)\le i^+(\chi,b), && \text{for }\ (\chi,b)\in(-\tau,\tau)\times(0,\infty).\end{aligned}$$

Moreover, the solution component $S(\chi)$ is monotonically decreasing,

$$S'(\chi)\le 0,\quad \text{for all }\ \chi\in(-\tau,\tau).$$

Proof. Define the closed convex set,

$$E \triangleq \left\{ (\hat{S}, i_0) \in \mathbb{D} \mid S^-(\chi) \leq \hat{S}(\chi) \leq S_0,\, i^-(\chi,0) \leq i_0(\chi) \leq i^+(\chi,0) \right\},$$

where $\mathbb{D} = C^1([-\tau,\tau]) \times C([-\tau,\tau])$. For each $(\hat{S}, i_0) \in E$, consider the coupled parabolic-elliptic system,

$$\begin{cases} cS'' - d_1 S'' - \kappa S + \kappa \hat{S} - M\left(\hat{S}, \int_0^\infty \varrho(b)h(b)i\, db\right) + \int_0^\infty \sigma(b)i(\chi,b)db, & \chi \in (-\tau,\tau), \\ \dfrac{\partial i}{\partial b} - d_2 \Delta_\chi i - c\dfrac{\partial i}{\partial \chi} = 0, & \chi \in (-\tau,\tau),\, b \in (0,\infty), \\ i(\chi,0) = i_0(\chi), & \chi \in [-\tau,\tau], \\ S(\pm\tau) = S^-(\pm\tau), & i(\pm\tau,b) = i^-(\pm\tau,b),\, b \in (0,\infty). \end{cases} \tag{13}$$

Here, $\kappa \geq NM^{-1}(S_0, H(i(\chi,b))) \exp^{\lambda_1 \tau}$ is chosen sufficiently large. Note that when,

$$i_0(\chi) = M\left(\hat{S}, \int_0^\infty \varrho(b)h(b)i(\chi,b)\right) db \qquad \text{and} \qquad \hat{S}(\chi) = S(\chi),$$

the system reduces to problem (12). Define the operator $D = (D_1, D_2) : E \to \mathbb{D}$ by,

$$\begin{aligned} D_1(\hat{S}, i_0) &\triangleq S(\chi), \\ D_2(\hat{S}, i_0) &\triangleq M\left(\hat{S}, \int_0^\infty \varrho(b)h(b)i(\chi,b))\, db\right), \end{aligned}$$

for $\chi \in [-\tau,\tau]$. A fixed point of D corresponds to a solution of (12).

We first verify $D(E) \subseteq E$. For any $(\hat{S}, i_0) \in E$, we have

$$i_0(\chi) \leq i^+(\chi,0) \qquad \text{and} \qquad i(\pm\tau,b) = i^-(\pm\tau,b) \leq i^+(\pm\tau,b).$$

By the comparison principle,

$$i(\chi,b) \leq i^+(\chi,b) \qquad \text{for all} \qquad (\chi,b) \in [-\tau,\tau] \times [0,\infty).$$

Similarly, Lemma 2.4 gives $i(\chi,b) \geq i^-(\chi,b)$.

To show $S(\chi) \leq S_0$, observe that,

$$\kappa\hat{S} - M\left(\hat{S}, \int_0^\infty \varrho(b)h(b)i(\chi,b)\right) db + \int_0^\infty \sigma(b)i(\chi,b)db \leq \kappa S_0,$$

which implies,

$$-d_1(S_0 - S)'' + c(S_0 - S)' + \kappa(S_0 - S) \geq 0, \quad \chi \in (-\tau,\tau). \tag{14}$$

Since $S(\pm\tau) \leq S_0$, we conclude $S(\chi) \leq S_0$ for $\chi \in (-\tau,\tau)$.

For the lower bound, note that $-M\left(S, \int_0^\infty \varrho(b)h(b)i(\chi,b)\right)db + \int_0^\infty \sigma(b)i(\chi,b)db$ is increasing in $S \geq 0$. Lemma 2.3 yields,

$$\begin{aligned}
&-d_1(S^-)'' + c(S^-)' + \kappa S^- - \kappa\hat{S} + M\left(\hat{S}, \int_0^\infty \varrho(b)h(b)i(\chi,b)\right)db - \int_0^\infty \sigma(b)i(\chi,b)db \\
&\quad \leq -d_1(S^-)'' + c(S^-)' + M\left(S^-(\chi), \int_0^\infty \varrho(b)h(b)i^+(\chi,b)\right)db - \int_0^\infty \sigma(b)i^-(\chi,b)db \leq 0.
\end{aligned}$$

Hence, $S(\chi) \geq S^-(\chi)$ for $\chi \in [-\tau,\tau]$. Therefore, $S^-(\chi) \leq D_1(\hat{S}, i_0) = S(\chi) \leq S_0$.

For D_2, we have

$$\begin{aligned}
D_2(\hat{S}, i_0) &\leq \frac{\partial M(S_0,0)}{\partial i}\int_0^\infty \varrho(b)h(b)i^+(\chi,b)db = i^+(\chi,0), \\
D_2(\hat{S}, i_0) &\geq M\left(S^-, \int_0^\infty \varrho(b)h(b)i(\chi,b)\right)db = i^-(\chi,0).
\end{aligned}$$

Thus, $D(E) \subset E$.

Next, we prove continuity of D. The function $i(\chi,b)$ can be expressed as,

$$i(\chi,b) = D_0(b)(i_0) + i^+(.,b),$$

where $\{D_0(b)\}_{b\geq 0}$ is the semigroup generated by $d_2\Delta_\chi - c\nabla_\chi$ with Dirichlet boundary conditions, and $i^+(.,b)$ satisfies,

$$\frac{\partial \tilde{i}}{\partial b} = d_2\Delta_\chi \tilde{i} - c\frac{\partial \tilde{i}}{\partial \chi}, \qquad i(\pm\tau,b) = i^-(\pm\tau,b), \qquad i(\chi,0) = 0,$$

for $(\chi,b) \in (-\tau,\tau)\times(0,\infty)$. For any $(\hat{S}_1, i_0^1), (\hat{S}_2, i_0^2) \in E$, we estimate,

$$\begin{aligned}
&\left|M\left(\hat{S}_1, \int_0^\infty \varrho(b)h(b)i_1(\chi,b)\right)db - M\left(\hat{S}_2, \int_0^\infty \varrho(b)h(b)i_2(\chi,b)\right)db\right| \\
&\quad\leq \left|M\left(\hat{S}_2, \int_0^\infty \varrho(b)\left[h(b)i_1(\chi,b) - h(b)i_2(\chi,b)\right]db\right)\right| + \left|M\left(\hat{S}_1 - \hat{S}_2, \int_0^\infty \varrho(b)h(b)i_1(\chi,b)\right)db\right| \\
&\quad\leq \frac{\partial M(S_0,0)}{\partial i}\int_0^\infty \varrho(b)h(b)\left|D_0(i_0^1 - i_0^2)(\chi)\right|db + M_h M^{-1}(S_0,0)\exp^{\lambda_1\tau}\left|\hat{S}_1 - \hat{S}_2\right|.
\end{aligned} \tag{15}$$

Define,

$$G(\chi) = M\left(S(\chi), \int_0^\infty \varrho(b)h(b)i(\chi,b)\right)db - \int_0^\infty \sigma(b)i(\chi,b)db.$$

Using the variation of constants formula, $S(\chi)$ can be written as,

$$\begin{aligned}
S(\chi) = &\frac{1}{d_1(\lambda_0-\lambda_2)}\int_0^\chi e^{\lambda_2(\chi-y)}\left(\kappa\hat{S}(y) - G(y)\right)dy \\
&+ \frac{1}{d_1(\lambda_0-\lambda_2)}\int_0^\chi e^{\lambda_1(\chi-y)}(\hat{S}(y) - G(y)) + Q_1e^{\lambda_0\chi} + Q_2e^{\lambda_2\chi},
\end{aligned} \tag{16}$$

where Q_1, Q_2 are determined by boundary conditions and

$$\lambda_{0,2} = \frac{c \pm \sqrt{c^2 + 4d_1\kappa}}{2d_1}.$$

Combining (15) and (16) shows D is continuous. To prove D is compact, note $D(E) \subset E$ implies uniform boundedness. Differentiating (16),

$$\begin{aligned}S'(\chi) &= \frac{\lambda_2}{d_1(\lambda_0-\lambda_2)}\int_0^{\chi} e^{\lambda_2(\chi-y)}\left(\kappa\hat{S}(y) - M(\hat{S}(y),\int_0^{\infty}\varrho(b)h(b)i(y,b)) + \int_0^{\infty}\sigma(b)i(y,b)db\right)dy\\ &\quad + \frac{\lambda_0}{d_1(\lambda_0-\lambda_2)}\int_0^{\chi} e^{\lambda_0(\chi-y)}\left(\kappa\hat{S}(y) - M(\hat{S}(y),\int_0^{\infty}\varrho(b)h(b)i(y,b)) + \int_0^{\infty}\sigma(b)i(y,b)db\right)dy\\ &\quad + Q_1\lambda_0 e^{\lambda_0\chi} + Q_2\lambda_2 e^{\lambda_2\chi}.\end{aligned}$$

Thus,

$$\begin{aligned}|S'(\chi)| &\le \frac{\lambda_2\kappa S_0}{d_1(\lambda_0-\lambda_2)}\int_0^{\chi} e^{\lambda_2(\chi-y)}dy + \frac{\lambda_0\kappa S_0}{d_1(\lambda_0-\lambda_2)}\int_0^{\chi} e^{\lambda_0(\chi-y)}dy + |Q_1|\lambda_0 e^{\lambda_0\chi} + |Q_2|\lambda_2 e^{\lambda_2\chi}\\ &\le \frac{\kappa S_0}{d_1(\lambda_0-\lambda_2)}e^{\lambda_2\tau} + |Q_1|\lambda_0 e^{\lambda_0\tau} + |Q_2|\lambda_2 e^{\lambda_2\tau}.\end{aligned}$$

Similarly,

$$|S''(\chi)| \le \frac{2\kappa S_0}{d_1} + |Q_1|\lambda_0^2 e^{\lambda_0\tau} + |Q_2|\lambda_2^2 e^{\lambda_2\tau} + \frac{\kappa S_0}{d_1(\lambda_0-\lambda_2)}\left(\lambda_0 e^{\lambda_0\tau} + \lambda_2 e^{\lambda_2\tau}\right).$$

Hence $D_1(E)$ is equicontinuous in $C^1([-\tau,\tau])$. By **(A)** and [11], the map $(S,i_0) \to G(\chi)$ is compact from $C([-\tau,\tau])$ to $C^1([-\tau,\tau])$. Since $\hat{S} \in C^1([-\tau,\tau])$, $D_2(E)$ is equicontinuous in $C([-\tau,\tau])$. The Arzelà-Ascoli theorem implies D is completely continuous.

By Schauder's fixed point theorem, there exists $(\hat{S}(\chi), i_0(\chi)) \in E$ with

$$D(\hat{S}(\chi), i_0(\chi,b)) = (\hat{S}(\chi), G(\chi)), \text{ for } \chi \in [-\tau,\tau].$$

Standard arguments as in [11] show $(S(\chi), i(\chi,b))$ is a classical solution of (12).

Finally, we prove $S'(\chi) \le 0$. From the first equation of (12), $d_1S'' - cS' \ge 0$, equivalent to $\left(\exp\left\{-\frac{c}{d_1}\chi\right\}S'(\chi)\right)' \ge 0$. Thus,

$$\exp^{-\frac{c}{d_1}\chi} S'(\chi) \le \exp^{-\frac{c}{d_1}\tau} S'(\tau), \quad \chi \in [-\tau,\tau].$$

Since $\tau > \tau_0 \ge \frac{1}{\gamma}\ln L$ and $S(\chi) \ge 0$, Lemma 2.3 gives $S'(\tau) \le 0$. Therefore $S'(\chi) \le 0$ for $\chi \in [-\tau,\tau]$, completing the proof. □

2.3 Extension to the entire coordinate axis

This subsection extends the result of Theorem 2.1 to the entire coordinate axis. Let $\{\tau_n\}_{n\ge1}$ be a positive sequence satisfying $\tau_n > \tau_0$ and $\tau_n \to +\infty$ as $n \to \infty$. For each n, suppose $(S_n(\chi), i_n(\chi,b))$

is a solution of system (12) with $\tau = \tau_n$, which can be expressed as,

$$\begin{cases} cS_n' = d_1 S_n'' - M\left(S_n, \int_0^\infty \varrho(b)h(b)i_n(\chi,b)db\right) + \int_0^\infty \sigma(b)i_n(\chi,b)db, & \chi \in (-\tau_n,\tau_n), \\ \dfrac{\partial i_n}{\partial b} = d_2\Delta_\chi i_n - c\dfrac{\partial i_n}{\partial \chi}, & \chi \in (-\tau_n,\tau_n),\ b \in (0,\infty), \\ i_n(\chi,0) = M\left(S_n(\chi), \int_0^\infty \varrho(b)h(b)i_n(\chi,b)db\right), & \chi \in [-\tau_n,\tau_n], \\ S_n(\pm\tau_n) = S^-(\pm\tau_n), \quad i_n(\pm\tau_n,b) = i^-(\pm\tau_n,b), & b \in (0,\infty). \end{cases} \tag{17}$$

To facilitate the continuation process, we establish several a priori estimates. For brevity, define the domains,

$$\mathcal{D}_n = (-\tau_n,\tau_n), \quad \Gamma_n = (-\tau_n,\tau_n) \times (0,\infty).$$

Constants independent of n are denoted by $C > 0$ and may vary to avoid notational redundancy.

Lemma 2.5. *There exist an integer n_0 and a constant $C > 0$ such that for all $n > n_0$,*

$$S_n'(\tau_n) \le 0, \quad \frac{\partial i_n}{\partial \chi}(\tau_n,b) \le 0, \tag{18}$$

$$\int_{\mathcal{D}_n} i_n(\chi,b)d\chi + d_2\int_0^\infty \frac{\partial i_n}{\partial \chi}(-\tau_n,b)db + d_1 S_n'(-\tau_n) \le C, \tag{19}$$

$$\int_0^\infty i_n(\chi,b)db \le C, \quad \textit{for } \chi \in \mathcal{D}_n. \tag{20}$$

Proof. To prove (18), note that Lemma 2.3 implies $S_n(\tau_n) = S^-(\tau_n) = 0$ for $\tau_n > \tau_0 \ge \frac{1}{\eta}\ln\frac{S_0}{Q}$. Since $S_n(\chi) \ge S^-(\chi) \ge 0$, it follows that $S_n'(\tau_n) \le 0$. Similarly, Lemma 2.4 yields $\partial_\chi i_n(\tau_n,b) \le 0$.

Define $\Psi_n(\chi,b) = \int_0^b i_n(\chi,s)ds$. Then, Ψ_n satisfies,

$$i_n(\chi,b) = d_2\partial_\chi^2\Psi_n(\chi,b) - c\partial_\chi\Psi_n(\chi,b) + M\left(S_n, \int_0^\infty \varrho(b)h(b)i_n(\chi,b)db\right). \tag{21}$$

Combining (21) with the first equation of (17) gives,

$$i_n(\chi,b) = d_2\partial_\chi^2\Psi_n(\chi,b) - c\partial_\chi\Psi_n(\chi,b) + d_1S_n'' - cS_n' + \int_0^\infty \sigma(b)i_n(\chi,b)db. \tag{22}$$

Integrating (22) over $\mathcal{D}_n$ and applying (18) yields,

$$\int_{\mathcal{D}_n} i_n(\chi,b)d\chi + d_2\partial_\chi\Psi_n(-\tau_n,b) + d_1S_n'(-\tau_n) \le c\int_0^b i^-(-\tau_n,s)ds + cS^-(-\tau_n),$$

which implies (19).

To establish (20), observe that for $\chi \in \left(-\frac{1}{\gamma}\ln L, \tau_n\right)$, the bound holds. For $\chi < -\frac{1}{\gamma}\ln L$, integrate (21) over $(-\tau_n,\chi)$ and use $i_n(\chi,0) \ge 0$ and (19) to obtain $c\Psi_n(\chi,b) \ge d_2\partial_\chi\Psi_n(\chi,b) - C$. This completes the proof. □

Lemma 2.6. *For sufficiently large n_1, there exists a constant B such that for $n \geq n_1$ and $(\chi, b) \in \Gamma_n$,*

$$i_n(\chi, b) \leq B, \quad \iint_{\Gamma_n} M\left(S_n(\chi), \varrho(b)h(b)i_n(\chi, b)\right) dbd\chi \leq B. \tag{23}$$

Proof. The proof follows the methodology in [13]. □

Lemma 2.7. *There exists a constant B such that for $n \geq n_2 \triangleq \max\{n_0, n_1\}$ and $\chi \in \mathcal{D}_n$,*

$$-B \leq S_n'(\chi) \leq 0, \quad |S_n''(\chi)| \leq B. \tag{24}$$

Proof. The result follows from the monotonicity of $S_n(\chi)$ and standard estimates. □

Lemma 2.8. *For $n \geq n_2$, there exists a constant Z such that,*

$$\begin{aligned} \iint_{\Gamma_n} \left[(\partial_\chi i_n)^2 + (\partial_b i_n)^2\right] dbd\chi + \int_{\mathcal{D}_n} \left[i_n^2 + (\partial_\chi i_n)^2\right] d\chi \leq Z, \\ \iint_{\Gamma_n} (\Delta_\chi i_n)^2 dbd\chi \leq Z. \end{aligned} \tag{25}$$

Proof. Note that, $i_n(\tau_n, b) = 0$, $i_n(-\tau_n, b) = i^-(-\tau_n, b)$, and $i_n(\chi, b) \geq i^-(\chi, b) \geq 0$. Thus,

$$\partial_b^2 i_n(-\tau_n, b) \geq \partial_b^2 i^-(-\tau_n, b).$$

Multiply the second equation of (17) by i_n and integrate over $\mathcal{D}_n$,

$$\begin{aligned} \frac{1}{2}\frac{d}{db}\int_{\mathcal{D}_n} i_n^2 d\chi + d_2 \int_{\mathcal{D}_n} (\partial_\chi i_n)^2 d\chi &= \frac{c}{2} i_n^2(-\tau_n, b) - d_2 i_n(-\tau_n, b)\partial_\chi i_n(-\tau_n, b) \\ &\leq \frac{c}{2}(i^-(-\tau_n, b))^2 - d_2 i^-(-\tau_n, b)\partial_\chi i^-(-\tau_n, b). \end{aligned} \tag{26}$$

From the initial condition and bounds (20) and (23),

$$\begin{aligned} \int_{\mathcal{D}_n} i_n^2(\chi, 0) d\chi &= \int_{\mathcal{D}_n} \left[M\left(S_n, \int_0^\infty \varrho(b)h(b)i_n db\right)\right]^2 d\chi \\ &\leq \|\varrho\|_\infty \|h\|_\infty C \iint_{\Gamma_n} M\left(S_n, \varrho(b)h(b)i_n\right) dbd\chi \leq Z. \end{aligned} \tag{27}$$

Integrating (26) over $(0, b)$ and using the form of $i^-(-\tau_n, b)$,

$$\frac{1}{2}\int_{\mathcal{D}_n} i_n^2(\chi, b) d\chi + d_2 \int_0^b \int_{\mathcal{D}_n} (\partial_\chi i_n)^2 d\chi ds \leq Z. \tag{28}$$

To prove the first part of (25), we also need,

$$\int_{\mathcal{D}_n} (\partial_\chi i_n)^2 d\chi + \iint_{\Gamma_n} (\partial_b i_n)^2 dbd\chi \leq Z. \tag{29}$$

Multiply the second equation of (17) by $\partial_b i_n$ and integrate over $(0, b) \times \mathcal{D}_n$,

$$\begin{aligned} &\int_0^b \int_{\mathcal{D}_n} (\partial_b i_n)^2 d\chi ds + \frac{d_2}{2}\int_{\mathcal{D}_n} (\partial_\chi i_n)^2 d\chi \\ &= \frac{d_2}{2}\int_{\mathcal{D}_n} (\partial_\chi i_n(\chi, 0))^2 d\chi - d_2 \int_0^b \partial_\chi i_n(-\tau_n, s)\partial_b i_n(-\tau_n, s) ds - c\int_0^b \int_{\mathcal{D}_n} \partial_\chi i_n \partial_b i_n d\chi ds. \end{aligned} \tag{30}$$

Note that,

$$\partial_\chi i_n(\chi,0) = \frac{\partial M}{\partial S} S_n' + \frac{\partial M}{\partial i}\int_0^\infty \varrho(b)h(b)\partial_\chi i_n db.$$

Using (20), (23), (24), (28), and Minkowski's inequality,

$$\|\partial_\chi i_n(\chi,0)\|_{L^2(\mathcal{D}_n)} \le N\left\|\int_0^\infty i_n db\right\|_{L^\infty(\mathcal{D}_n)}\int_0^\infty \|i_n\|_{L^1(\mathcal{D}_n)}db + M(S_0,\|\partial_\chi i_n\|_{L^2(\Gamma_n)}) \le Z. \quad (31)$$

Also,

$$-d_2\int_0^b \partial_\chi i_n(-\tau_n,s)\partial_b i_n(-\tau_n,s)ds \le \frac{1}{2}\int_0^\infty (\partial_\chi i^-)^2 ds + \frac{d_2}{2}\int_0^\infty (\partial_b i^-)^2 ds \le Z.$$

The last term in (30) is bounded by Cauchy-Schwarz,

$$\begin{aligned}
&-c\int_0^b\int_{\mathcal{D}_n}\partial_\chi i_n \partial_b i_n d\chi ds\\
&\le \frac{1}{2}\int_0^b\int_{\mathcal{D}_n}(\partial_b i_n)^2 d\chi ds + \frac{c^2}{2}\int_0^b\int_{\mathcal{D}_n}(\partial_\chi i_n)^2 d\chi ds \le \frac{1}{2}\int_0^b\int_{\mathcal{D}_n}(\partial_b i_n)^2 d\chi ds + Z.
\end{aligned}$$

This establishes the first inequality in (25). The second follows by integrating the second equation of (17) and using (28), (29), and mean-value estimates. □

With these preparations, we now state the main result.

Theorem 2.2. *Assume $R_0 > 1$ and $c > c^*$. Then, system (4) admits a nontrivial bounded positive solution $(S(\chi), i(\chi,b))$ satisfying condition (5) with $S_0 > S^* \ge 0$. Moreover, $S \in W^{2,\infty}(\mathbb{R})$ is decreasing, and*

$$\lim_{\chi\to\infty} i(\chi,b)e^{-\lambda_1\chi} = \exp(-\varpi^* b),\ \forall\ b\in[0,\infty).$$

Proof. By the lemmas in Subsection 2.1, we extract a subsequence (S_n, i_n) converging to (S,i) in the following topologies;

$$\begin{aligned}
&S_n \to S \quad \text{in } C^1_{loc}(\mathbb{R}),\\
&i_n \to i \quad \text{in } L^2_{loc}(\Gamma), \quad \text{a.e.},\\
&i_n \rightharpoonup i \quad \text{in } H^1_{loc}(\Gamma), \quad L^2_{loc}(H^2_{loc}(\Gamma),(0,\infty)),
\end{aligned} \quad (32)$$

where $\Gamma = \mathbb{R}\times(0,\infty)$. Fatou's lemma and weak convergence imply,

$$i \in H^1(\Gamma)\cap L^2(H^2(\Gamma),(0,\infty)), \quad S' \in L^\infty(\mathbb{R}). \quad (33)$$

To verify that (S,i) solves (4), note that by uniform convergence and dominated convergence,

$$M\left(S_n, \int_0^\infty \varrho(b)h(b)i_n db\right) \to M\left(S, \int_0^\infty \varrho(b)h(b)i db\right), \quad \chi\in\mathbb{R}.$$

Thus, S satisfies,

$$cS' = d_1 S'' - M\left(S, \int_0^\infty \varrho(b)h(b)idb\right) + \int_0^\infty \sigma(b)idb, \quad \chi\in\mathbb{R}.$$

From (32), $S'' \in L^\infty(\mathbb{R})$. For any $\psi \in C_0^\infty(\mathbb{R} \times (0,\infty))$, for large n,

$$\int_{\mathbb{R}} i_n(\chi,0)\psi(\chi,0)d\chi + \iint_\Gamma i_n \partial_b \psi db d\chi = d_2 \iint_\Gamma \partial_\chi i_n \partial_\chi \psi db d\chi + c \iint_\Gamma \psi \partial_\chi i_n db d\chi.$$

Passing to the limit,

$$\iint_\Gamma \psi \left(\partial_b i - d_2 \partial_\chi^2 i + c\partial_\chi i\right) db d\chi = 0,$$

so, $\partial_b i = d_2 \partial_\chi^2 i - c\partial_\chi i$ in $\mathcal{D}'(\mathbb{R} \times (0,\infty))$. The boundary condition $i(\chi,0) = M\left(S, \int_0^\infty \varrho(b)h(b)idb\right)$ holds a.e.

Since $i\left(\chi,0\right) \in W^{2,2}(\mathbb{R})$, Sobolev embedding and L^p-estimates imply (S,i) is a classical solution.

For asymptotic behavior, lemmas in Subsection 2.1 ensure,

$$\lim_{\chi\to\infty} S(\chi) = S_0, \quad \lim_{\chi\to\infty} i(\chi,b) = 0,$$

uniformly in b. Since $S' \le 0$ and $S \in W^{2,\infty}(\mathbb{R})$, there exists $S^* = \lim_{\chi\to-\infty} S(\chi)$ with $0 \le S^* \le S_0$. If $S^* = S_0$, then $\int_0^\infty \varrho(b)h(b)idb \equiv 0$, implying $i \equiv 0$, which contradicts $i \ge i^- \ge 0$. Hence, $S^* < S_0$.

To show $i(+\infty,b) = 0$, take a sequence $\{\chi_n\} \to +\infty$ and define $\phi_n(\chi,b) = i(\chi+\chi_n,b)$. By (33), $\{\phi_n\}$ is bounded in $C^1(D\times(0,\infty))$ for $D = (0,\sigma)$. Extract a subsequence with $\phi_n \to \phi$ in $C_{loc}(D\times(0,\infty))$. Then,

$$\int_{\chi_n}^{\chi_{n+1}} i(\chi,b)d\chi \ge \int_D \phi_n(\chi,b)d\chi.$$

Since $\sum_{n=0}^{\infty} \int_{\chi_n}^{\chi_{n+1}} i(\chi,b)d\chi \le \int_{\mathbb{R}} i(\chi,b)d\chi \le C$, we have

$$\lim_{n\to\infty} \int_{\chi_n}^{\chi_{n+1}} i(\chi,b)d\chi = 0.$$

By dominated convergence, $\int_D \phi_n d\chi \to \int_D \phi d\chi = 0$, so $\phi \equiv 0$, i.e., $i(+\infty,b) = 0$.

Finally, since $i^- \le i \le i^+$,

$$\lim_{\chi\to-\infty} \left|i(\chi,b)e^{-\lambda_1\chi} - \exp(-\varpi^* b)\right| \le Le^{-\eta_1 b} \lim_{\chi\to-\infty} e^{\gamma\chi} = 0,$$

which completes the proof. □

3 Nonexistence of Traveling Waves

Theorem 3.1. *Assume that $R_0 \le 1$. Then, for any positive wave speed c, the system admits only the disease-free equilibrium solution, i.e.,*

$$S(\chi) \equiv S_0, \qquad i(\chi,b) \equiv 0,$$

for the system;

$$\begin{aligned} \frac{\partial i}{\partial b} &= d_2 \Delta_\chi i - \frac{\partial i}{\partial \chi}, \\ i(\chi, 0) &= \frac{\partial M(S_0 - \sigma, \sigma)}{\partial i} \int_0^\infty \rho(b) h(b) i(\chi, b) db. \end{aligned} \tag{34}$$

Proof. From the comparison principle applied to the second equation of system (4), we obtain

$$\sup_{\chi \in \mathbb{R}} \|i(\chi, b)\| \leq \|i(\chi, 0)\|_\infty.$$

By considering the boundary conditions and assumption (M), it follows that,

$$\|i(\chi, 0)\|_\infty \leq \frac{\partial M(S_0, 0)}{\partial i} \int_0^\infty \rho(b) h(b)\, db\, \|i(\chi, 0)\|_\infty \triangleq R_0 \|i(\chi, 0)\|_\infty.$$

When $R_0 < 1$, this inequality implies that $i(\chi, b) \equiv 0$.

Substituting $i(\chi, b) \equiv 0$ into the first equation of system (4), we obtain

$$e^{-\frac{c}{d_1}\chi} S'(\chi) \leq \int_0^\infty \rho(b) I(\chi, b)\, db,$$

where $\vartheta \leq 0$. If $\vartheta < 0$, integrating both sides from $-\infty$ to $+\infty$ yields,

$$S_0 - S_1 = \int_{-\infty}^{+\infty} f\, e^{-\frac{c}{d_1}\chi}\, d\chi. \tag{35}$$

The left-hand side of (35) is nonnegative, while the right-hand side tends to $-\infty$, which is a contradiction. Therefore, $\vartheta = 0$ and hence, $S'(\chi) = 0$, implying that $S(\chi) \equiv S_0$.

It remains to consider the case $R_0 = 1$. From (23), we have $i(\chi, b) \leq J$ for some positive constant J. Let χ_0 be a point where,

$$i(\chi_0, 0) = \max\{C, \|i(\chi, 0)\|_\infty\} \geq 0.$$

If $\chi_0 = \pm\infty$, then it is immediate that $i(\chi, b) \equiv 0$. Otherwise, the boundary conditions imply that $\chi_0 \in [-n, n] \subset \mathbb{R}$ for some finite n. Evaluating the second equation of (4) at $\chi = \chi_0$ gives,

$$i(\chi_0, 0) \leq \frac{M(S(\chi_0), H(i(\chi_0, 0)))}{M(S_0, 0)}\, i(\chi_0, 0),$$

which implies that $M(S(\chi_0), H(i(\chi_0, 0))) = M(S_0, 0)$. Since $S'(\chi) \leq 0$ and $M(S, i)$ is continuous, we conclude that $S(\chi) \equiv S_0$. Substituting this into the first equation of system (4) yields,

$$M\left(S_0, \int_0^\infty \rho(b) h(b)\, i(\chi, b)\, db\right) = 0.$$

By assumption **(H)**, it follows that $i(\chi, b) \equiv 0$. Hence, the theorem is proved. □

Theorem 3.2. *Set $R_0 > 1$ and $c \in (0, c^*)$, then system (4) has no nontrivial (TWS).*

Proof. Conversely, suppose that system (4) has a constrained positive solution such that,

$$(S(-\infty), i(-\infty, b)) = (S_0, 0). \tag{36}$$

For every $0 < \nu_\sigma^* \leq \nu^*$ may be found for any $0 \leq \sigma < \min\{S_0, \zeta\} \triangleq \delta$ such that,

$$\frac{\partial M(S_0 - \sigma, \sigma)}{\partial i} \int_0^\infty \rho(b)h(b)\exp\{-\nu_\sigma^* b\}db = 1.$$

In light of (36), there is a sufficiently big constant M_σ such that, for $\chi < -M_\sigma$, we obtain

$$S_0 - \sigma \leq S(\chi) \leq S_0 \qquad \text{and} \qquad 0 < i(\chi, b) \leq \zeta.$$

Thus, $\vartheta = \dfrac{2\pi c_0}{\Gamma_c} \in \mathbb{Q}_+$ and $c_0 \in (0, c)$ are selected for convenience. We may use the following formula,

$$g_c(\chi, b) \triangleq \exp^{\frac{\bar{c}\chi}{2d_2}} \sin\left(\frac{\Gamma_{\bar{c}}\chi}{d_2}\right) \exp^{-\nu_\sigma^* b},$$

where $\bar{c} \in (0, c]$ and $\Gamma_{\bar{c}} = \sqrt{4d_2\nu_\sigma^* - \dfrac{\bar{c}^2}{2}}$. Hence, $g_c(\chi, b)$ satisfies (34).

A sufficiently large constant M_σ exists in light of (36) such that for $\chi < -M_\sigma$, we get

$$S_0 - \sigma \leq S(\chi) \leq S_0 \qquad \text{and} \qquad 0 < i(\chi, b) \leq \zeta.$$

For convenience, set $\vartheta = \dfrac{2\pi c_0}{\Gamma_c} \in \mathbb{Q}_+$ and $c_0 \in (0, c)$. The following formula is thus available to us,

$$\frac{\vartheta}{2}(2u + 2p_0 + 1) \geq (u + p_0)\vartheta + 1 \geq (u + p_0)\vartheta \geq \frac{\vartheta}{2}(2u + 2p_0 - 1),$$

where $u \in \mathbb{N}$ and $p_0 \in \mathbb{N}$ are fixed. Moreover, we may find appropriate p_0, $u \in \mathbb{N}$ such that,

$$\frac{-(2p_0 - 1)d_2\pi}{\Gamma_c} \leq -M_\sigma \qquad \text{and} \qquad p \triangleq (u + p_0)\vartheta \in \mathbb{N}.$$

Thus, $q_1 \leq \chi_1 \leq \chi_2 \leq q_2 \leq -M_\sigma$, where

$$q_1 = \frac{(2u + 2p_0 + 1)d_2\pi}{\Gamma_c}, \qquad q_2 = \frac{(2j + 2p_0 - 1)d_2\pi}{\Gamma_c},$$
$$\chi_1 = \frac{(p + 1)d_2\pi}{\Gamma_{c_0}}, \qquad \chi_2 = \frac{pd_2\pi}{\Gamma_{c_0}}.$$

Since $\sin(\Gamma_{c_0}\chi/d_2) = 0$ $(l = 1, 2)$ and $\sin(\Gamma_{c_0}\chi/d_2) = 0$ for $\chi \in [\chi_1, \chi_2]$, we may conclude that $\sin(\Gamma_{c_0}\chi/d_2) > 0$. Therefore, $\zeta_1 > 0$ exists such that,

$$\zeta_1 g_{c_0}(\chi, b) \leq g_c(\chi, b) \quad \text{and} \quad g_{c_0}(\chi_1, 0) = 0, \quad \text{for} \quad \chi > 0, \quad b \in [\chi_1, \chi_2].$$

There exists $\epsilon_2 > 0$ such that $\zeta_2 g_c(\chi, 0) \leq i(\chi, 0)$ for $\chi \in [x_1, x_2]$ and $g_c(x_1, 0) = 0$, given that $i(\chi, 0) > 0$. The comparison principle implies that,

$$\epsilon_1\epsilon_2 g_{c_0}(\chi, b) \leq i(\chi, b), \quad \text{for} \quad b > 0, \quad x \in [x_1, x_2].$$

For $b > 0$ and $\chi \in [\chi_1, \chi_2]$, $\zeta_1\zeta_2 g_{c_0}(\chi, b) \leq i(\chi, b)$. Hence $\dot{i}(t, x, b)$ satisfies $\triangleq i(\chi - (c_0 - c)t, b)$, we assert that $\zeta_1\zeta_2 g_{c_0}(\chi, b)$ is a lower solution of $\dot{i}(t, \chi, b)$.

Thus, by changing the equation above to $v(t,\chi,b) = \tilde{i}(t,x,b) - \zeta_1\zeta_2 g_{c_0}(\chi,b)$, we really get,

$$\begin{cases} \dfrac{\partial v}{\partial b} + \dfrac{\partial v}{\partial t} = d_2\Delta_x v - c\dfrac{\partial v}{\partial \chi}, \\ v(t,x,0) \geq \dfrac{\partial M(S_0-\sigma,\sigma)}{\partial i} \displaystyle\int_0^\infty \rho(b)h(b)v(t,x,b)\,db. \end{cases} \tag{37}$$

Using the maximum principle [13], we find that $v(t,x,b) \geq 0$ for $t \geq 0$, $b \geq 0$, and $x \in [q_1,q_2]$. Consequently,

$$\zeta_1\zeta_2 g_{c_0}(\chi,b) \leq i(x+(c_0-c)t,b), \quad t \geq 0,\ b \geq 0 \ \text{ and } \ x \in [q_1,q_2].$$

Inequality $c_0 < c$ implies that the right side tends to zero as $t \to \infty$, while the left side is greater than zero as $x \in (q_1,q_2)$, which ensures our result. □

4 Numerical Simulation

In this section, focusing on TWS of system (3), we perform some numerical simulations. Therefore, we assume the following parameters $\rho(b) = 0.3 + 0.1b$, $\mu(b) = 0.1 + 0.2b$, $\alpha = 0,1$, and $\varrho(a) = 0.1 + 0.1b$.

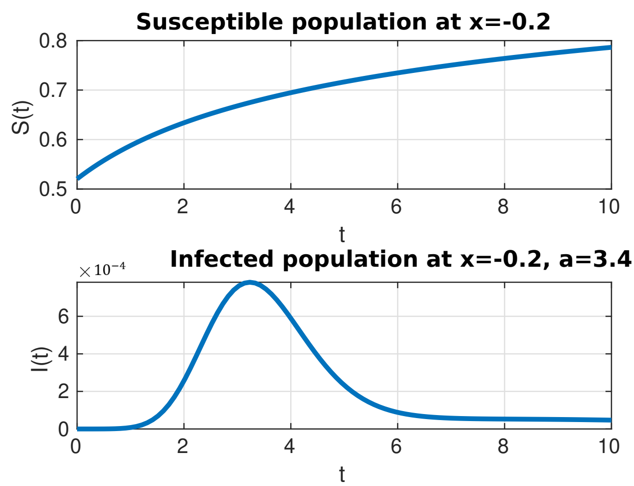

Figure 1: The solution of the system (3) at x and a fixed, where $M(S,I) = \dfrac{SI}{1+\alpha I}$.

Figure 1 tracks the temporal evolution of S and I at a fixed spatial point and infection age, illustrating the transition from the initial perturbation to the long-term epidemiological regime. Consistent with theoretical predictions for age-structured SIS systems with diffusion, the susceptible population initially decreases due to infection pressure while the infected population rises until the balance between transmission and recovery is established. Over time, the trajectories approach the endemic equilibrium levels predicted by the traveling wave analysis, confirming that the local temporal dynamics align with the wave profile as it passes through the spatial location.

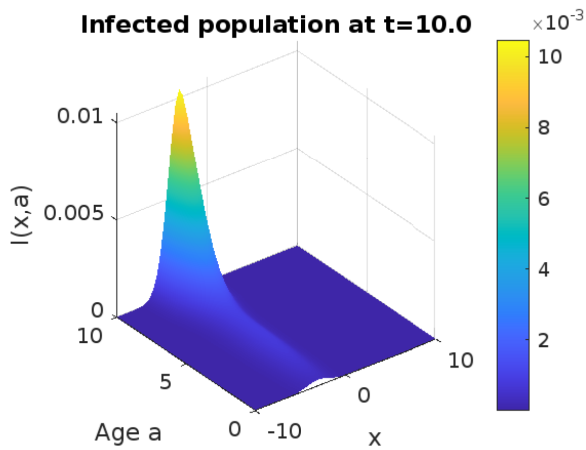

Figure 2: The infected population at fixed time $t = 10$ and different position x values ensure that a TWS exists. .

Figure 2 presents a three-dimensional surface representation of the infected population over space and age at the terminal simulation time, clearly demonstrating the joint spatiotemporal age dynamics of disease spread within the model.

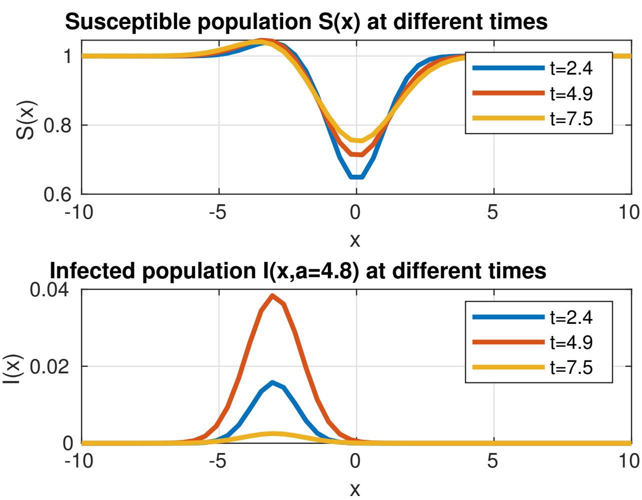

Figure 3: TWS of (3) at different time t where $M(S,I) = \dfrac{SI}{1+\alpha I}$ and with $\rho(b) = 0.3 + 0.1b$, $\mu(b) = 0.1 + 0.2a$, $\alpha = 0, 1$, and $\rho(b) = 0.1 + 0.1b$.

Figure 3 shows the spatial distribution of susceptible (S) and infected (I) populations at different times for a fixed age, revealing how diffusion and infection waves propagate through space.

The theoretical minimal wave speed c^* thus plays a significant role in epidemiology, as it characterizes the threshold rate at which an infection can spatially invade a population of susceptibles. Concretely, if public-health interventions like mobility restrictions, vaccination, or changes in behavior lower the effective transmission and movement parameters sufficiently, so that the resulting wave speed falls below c^*, spatial spread is not sustainable, and the epidemic dies out in the long run. This result ties wave theory to real-world control policies for recurrent infections with spatial transmission like COVID-19, malaria, and other reinfection-driven diseases. Additionally, the analytical results on c^* give insight into the minimum spatial response required for containment policy in environments characterized by the potential for reinfection, heterogeneity of contact structure, or behavioral adaptation.

5 Conclusion

This paper has extended classical SIS traveling-wave theory by coupling a generalized nonlinear incidence operator with an infection-age structured infected class, and by analyzing the resulting integro-differential PDE system to establish the existence of traveling waves and characterize the minimal wave speed c^*. In comparison with the earlier age-structured SIS works such as [12, 14] and recent fractional and data-informed epidemic studies [16, 15], the major contributions of this paper are: explicitly incorporating infection-age dependent infectiousness inside a generalized incidence operator in a diffusive spatial setting; developing proper upper and lower solution frameworks adapted to age-dependent damping and nonlinearity; and rigorously characterizing c^* in terms of the model's spectral and integral operators, highlighting how age-dependent infectivity and nonlinear interaction jointly influence wave propagation. Where in this paper, R_0 is a critical threshold value to ascertain whether or not TWS , as demonstrated by theoretical studies. In particular, we demonstrated that there are traveling wave solutions for all wave speeds $c > c^*$, where $c^* > 0$ is the minimum wave speed, whenever $R_0 > 1$.

In contrast, when $c < c^*$ or when $R_0 \leq 1$, there are no such solutions. We proved that these waves exist for general nonlinear incidence functions by using an iterative process to create appropriate upper and lower solutions. The theoretical results were further confirmed by numerical simulations, which clearly showed how the illness traveled over space. By adding more nonlinearity to our system by taking into account a nonlinear incidence function we want to further generalized our methodology in our next work. We anticipate that this will broaden our comprehension of the underlying dynamics and present new difficulties in proving the existence of traveling wave solutions. Future work can also extend this framework to include fractional order derivatives representing memory dependent transmission and long-range effects in time, which have recently been considered in the context of fractional epidemic models. Another promising approach involves adding stochastic perturbations, standing for random environmental or demographic fluctuations with possible effects on epidemic persistence and wave dynamics.

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Conflicts of Interest The authors declare no conflict of interest.

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